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2000 J. Phys. A: Math. Gen. 33 3533

(http://iopscience.iop.org/0305-4470/33/18/301)

J. Phys. A: Math. Gen. **33** (2000) 3533–3543. Printed in the UK PII: S0305-4470(00)10311-7

Size and area of square lattice polygons

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Received 10 November 1999

Abstract. We use the finite lattice method to calculate the radius of gyration, the first and second area-weighted moments of self-avoiding polygons on the square lattice. The series have been calculated for polygons up to perimeter 82. Analysis of the series yields high accuracy estimates confirming theoretical predictions for the value of the size exponent, $\nu = \frac{3}{4}$, and certain universal amplitude combinations. Furthermore, a detailed analysis of the asymptotic form of the series coefficients provide the firmest evidence to date for the existence of a correction-to-scaling exponent, $\Delta = \frac{3}{2}$.

1. Introduction

A self-avoiding polygon (SAP) can be defined as a walk on a lattice which returns to the origin and has no other self-intersections. The history and significance of this problem is nicely discussed in [1]. Generally, SAPs are considered distinct up to translations, so if there are p_n SAPs of perimeter length n there are $2np_n$ walks (the factor of two arising since the walk can go in two directions). In addition to enumerations by perimeter, one can also enumerate polygons by the enclosed area (or number of unit cells), or both perimeter and area. Of particular interest are the first few area-weighted moments of the perimeter generating function. Also of great interest is the mean-square radius of gyration, which measures the typical size of a SAP.

This paper builds on the work of Enting [2] who used transfer matrix techniques to enumerate square lattice polygons by perimeter to 38 steps. This enumeration was later extended by Enting and Guttmann to 46 steps [3] and then to 56 steps [4]. This latter work also included calculations of moments of the caliper size distribution. Hiley and Sykes [5] obtained the number of square lattice polygons by both area and perimeter up to perimeter 18. Enting and Guttmann extended the calculation to perimeter 42 [6]. The radius of gyration was calculated for SAPs up to 28 steps by Privman and Rudnick [7], using a technique based on direct counting of compact site animals on the dual lattice. Recently, Jensen and Guttmann devised an improved algorithm for the enumeration of SAPs and extended the calculation to 90 steps [8]. The work reported here is based on generalizations of this improved algorithm. This has enabled us to extend the calculation of the radius of gyration and the first two area-weighted moments to 82 step SAPs. The generalization of the transfer matrix technique to area-weighted moments is similar to the one used by Conway [9] in his calculation of series for percolation problems and lattice animals. The generalization to the radius of gyration has to our knowledge no counterpart in the published literature, and represents a major advance in the design of efficient counting algorithms. Previous calculations of the radius of gyration were based on direct counting algorithms. The transfer matrix algorithm used in this paper is exponentially faster and thus enables us to significantly extend the series (see [8] for further details).

The size exponent, ν, for SAPs is believed to be identical to that of self-avoiding walks (SAWs). This has been argued theoretically from the connection between the energy–energy and spin–spin correlation functions of the n-vector model in the limit $n \to 0$, and SAPs and SAWs, respectively [10, 11]. Alternatively, it has also been obtained from real space renormalization group arguments [12]. The exponent describing the growth of the mean area of polygons of perimeter n is expected to be 2ν [13]. Intuitively this is not surprising since it just means that the average area of a polygon is proportional to the square of the radius of gyration. So one is merely finding that for this problem the typical area and typical length scale match one another nicely. These expectations have been confirmed reasonably accurately by numerical work [4, 6, 7].

The functions we consider in this paper are: (i) the polygon generating function, $\mathcal{P}(u) = \sum p_n u^n$; (ii) kth area-weighted moments of polygons of perimeter n, $\langle a^k \rangle_n$; and (iii) the mean-square radius of gyration of polygons of perimeter n, $\langle R^2 \rangle_n$. These quantities are expected to behave as

$$\begin{aligned} p_n &= B\mu^n n^{\alpha-3}[1 + \mathrm{o}(1)] \\ \langle a^k \rangle_n &= E^{(k)} n^{2k\nu}[1 + \mathrm{o}(1)] \\ \langle R^2 \rangle_n &= D n^{2\nu}[1 + \mathrm{o}(1)] \end{aligned} \tag{1}$$

where $\mu = u_c^{-1}$ is the reciprocal of the critical point of the generating function, and $\alpha = \frac{1}{2}$ and $\nu = \frac{3}{4}$ are known exactly [14], though non-rigorously. It is also known [15] that the amplitude combination $E^{(1)}/D$ is universal, and that

$$BD = \frac{5}{32\pi^2}\sigma a_0 \tag{2}$$

where a_0 is the area per site and σ is an integer such that p_n is non-zero only if n is divisible by σ. For the square lattice $a_0 = 1$ and $\sigma = 2$. These predictions have been confirmed numerically [15, 16].

In the next section we describe the generalization of the finite lattice method required in order to calculate the radius of gyration and area-weighted moments of SAPs. The results of the analysis of the series are presented in section 3.

2. Enumeration of SAPs

The method used to enumerate SAPs in this work is based on the method devised by Enting [2] for enumerations by perimeter and uses the enhancements of Jensen and Guttmann [8]. In the following we first very briefly outline the original method and then show how to generalize it in order to calculate area-weighted moments and the radius of gyration. Details of the algorithm can be found in the papers cited above.

The first terms in the series for the perimeter generating function can be calculated using transfer matrix techniques to count the number of polygons spanning (in both directions) rectangles $W + 1$ edges wide and $L + 1$ edges long. The transfer matrix technique involves drawing a boundary through the rectangle intersecting a set of $W + 2$ edges. For each configuration of occupied or empty edges along the boundary we maintain a (perimeter) generating function for partially completed polygons. Polygons in a given rectangle are enumerated by moving the boundary so as to add one site at a time. Due to the obvious symmetry of the lattice one need only consider rectangles with $L \geqslant W$. Any polygon spanning such a rectangle has a perimeter of length at least $2(W+L)$. By adding the contributions from all rectangles of width $W \leqslant W_{\max}$ (where the choice of $W_{\max}$ depends on available computational resources) and length $W \leqslant L \leqslant 2W_{\max} - W + 1$, with contributions from rectangles with $L > W$ counted twice, the number of polygons per vertex of an infinite lattice is obtained correctly up to perimeter $n_{\max} = 4W_{\max} + 2$. The number of configurations required as $W_{\max}$ is increased grows exponentially as $\lambda^{W_{\max}}$, where $\lambda \simeq 2$ for the improved algorithm [8]. In addition to the dominant exponential growth in memory requirements there is a prefactor, which is proportional to the number of terms $n_{\max}$.

2.1. Area-weighted moments

Area-weighted moments can easily be calculated from the perimeter and area generating function

$$\mathcal{C}(u, v) = \sum_{n,m} c_{n,m} u^n v^m \tag{3}$$

where $c_{n,m}$ is the number of polygons with perimeter n and area m. From this we get the area-weighted generating functions,

$$\mathcal{P}^{(k)}(u) = \left(v\frac{\partial}{\partial v}\right)^k \mathcal{C}(u, v)|_{v=1} = \sum_n \sum_m m^k c_{n,m} u^n = \sum_n p_n^{(k)} u^n \tag{4}$$

and we define the average moments of area for a polygon with perimeter n:

$$\langle a^k \rangle_n = p_n^{(k)} / p_n^{(0)} = \sum_m m^k c_{n,m} / p_n. \tag{5}$$

In order to calculate the moments of area through this approach we need to calculate a full two-parameter generating function, which generally will require a lot of computer memory. If we are only interested in the first few moments there is a much more efficient approach [9]. We simply replace the variable v by $1 + z$ thus obtaining the function

$$F(u, z) = \sum_{n,m} c_{n,m} u^n (1 + z)^m = \sum_{n,m} \sum_{k=0}^{m} \binom{m}{k} c_{n,m} u^n z^m. \tag{6}$$

Let, $F_i(u)$, be the coefficient of z^i in $F(u, z)$. Then we see that

$$F_0(u) = \sum_{n,m} c_{n,m} u^n = \mathcal{P}(u)$$

$$F_1(u) = \sum_{n,m} m c_{n,m} u^n = \mathcal{P}^{(1)}(u)$$

$$F_2(u) = \sum_{n,m} m(m-1)/2 c_{n,m} u^n = [\mathcal{P}^{(2)}(u) - \mathcal{P}^{(1)}(u)]/2$$

and so on. Thus if we are only interested in the first and second moments of area we can truncate the series $F(u, z)$ at second order in z and find the relevant moments as $\mathcal{P}^{(1)}(u) = F_1(u)$ and $\mathcal{P}^{(2)}(u) = 2F_2(u) + F_1(u)$. The growth in memory requirements is still dominated by the exponential growth in the number of configurations. However, we have managed to turn the calculation of these moments from a problem with a prefactor cubic in $W_{\max}$ (the area is proportional to $W_{\max}^2$) into a problem with a prefactor linear in $W_{\max}$.

2.2. Radius of gyration

In the following we show how the definition of the radius of gyration can be expressed in a form suitable for a transfer matrix calculation. Note that we define the radius of gyration according to the *vertices* of the SAP and that the number of vertices equals the perimeter length. The radius of gyration of n points at positions $\boldsymbol{r}_i$ is

$$n^2 R_n^2 = \sum_{i>j} (\boldsymbol{r}_i - \boldsymbol{r}_j)^2 = (n-1) \sum_i (x_i^2 + y_i^2) - 2 \sum_{i>j} (x_i x_j + y_i y_j). \tag{7}$$

This last expression is suitable for a transfer matrix calculation. As usual [7] we actually calculate the generating function, $\mathcal{R}_g^2(u) = \sum_n p_n \langle R^2 \rangle_n n^2 u^n$. In order to do this we have to maintain five partial generating functions for each possible boundary configuration σ, namely:

- $P(u)$, the number of (partially completed) polygons according to perimeter.
- $R^2(u)$, the sum over polygons of the squared components of the distance vectors.
- $X(u)$, the sum of the x-component of the distance vectors.
- $Y(u)$, the sum of the y-component of the distance vectors.
- $XY(u)$, the sum of the 'cross' product of the components of the distance vectors, e.g., $\sum_{i>j}(x_i x_j + y_i y_j)$.

As the boundary line is moved to a new position each boundary configuration σ might be generated from several configurations σ' in the previous boundary position. The partial generation functions are updated as follows:

$$\begin{aligned}
P(u, \sigma) &= \sum_{\sigma'} u^{n(\sigma')} P(u, \sigma') \\
R^2(u, \sigma) &= \sum_{\sigma'} u^{n(\sigma')} [R^2(u, \sigma') + \delta(x^2 + y^2) P(u, \sigma')] \\
X(u, \sigma) &= \sum_{\sigma'} u^{n(\sigma')} [X(u, \sigma) + \delta x P(u, \sigma')] \\
Y(u, \sigma) &= \sum_{\sigma'} u^{n(\sigma')} [Y(u, \sigma) + \delta y P(u, \sigma')] \\
XY(u, \sigma) &= \sum_{\sigma'} u^{n(\sigma')} [XY(u, \sigma') + \delta x X(u, \sigma') + \delta y Y(u, \sigma')]
\end{aligned} \tag{8}$$

where $n(\sigma')$ is the number of occupied edges added to the polygon and $\delta = \min(n(\sigma'), 1)$.

2.3. Further particulars

Finally a few remarks of a more technical nature. The number of contributing configurations becomes very sparse in the total set of possible states along the boundary line and as is standard in such cases one uses a hash-addressing scheme. Since the integer coefficients occurring in the series expansions become very large, the calculation was performed using modular arithmetic. Up to eight primes were needed to represent the coefficients correctly. Further details and references are given in [8]. The series for the radius of gyration and area-moments were calculated for SAPs with perimeter length up to 82. The maximum memory required for any given width did not exceed 2 Gb. The calculations were performed on an eight node Alpha Server 8400 with a total of 8 Gb memory. The total CPU time required was about three days per prime. Obviously the calculation for each width and prime are totally independent and several calculations were done simultaneously.

In table 1 we have listed the series for the radius of gyration and first and second area-weighted moments. The series for the radius of gyration of course agree with the terms up to length 28 computed previously [7], while the terms up to length 40 for the first area moment agree with the series in [6]. The number of polygons of length $\leqslant 56$ can be found in [4] while those up to length 90 were reported in [8].

Table 1. The mean-square radius of gyration, first and second area-moments of n-step SAPs on the square lattice. Only non-zero terms are listed.

n	$p_n n^2 \langle R^2 \rangle_n$	$p_n \langle a \rangle_n$	$p_n \langle a^2 \rangle_n$
4	8	1	1
6	66	4	8
8	600	22	70
10	5 164	124	560
12	42 872	726	4 358
14	346 828	4 352	33 160
16	2 754 056	26 614	248 998
18	21 549 780	165 204	1 851 040
20	166 626 744	1 037 672	13 655 432
22	1 275 865 332	6 580 424	100 126 648
24	9 690 096 824	42 062 040	730 548 788
26	73 090 383 120	270 661 328	5 308 524 968
28	548 064 459 968	1 751 614 248	38 442 000 664
30	4 088 719 617 824	11 391 756 176	277 565 593 032
32	30 367 415 294 800	74 406 502 814	1 999 068 564 026
34	224 659 143 155 964	487 838 450 116	14 365 917 755 936
36	1 656 259 765 448 200	3 209 229 661 682	103 038 218 758 426
38	12 172 580 326 973 688	21 175 301 453 040	737 765 745 264 544
40	89 212 147 340 159 520	140 097 533 633 112	5 274 413 814 993 896
42	652 183 776 123 444 404	929 160 187 609 096	37 655 943 519 835 560
44	4 756 877 451 862 073 312	6 176 075 676 719 784	268 506 373 782 824 280
46	34 623 252 929 242 595 840	41 135 052 992 574 928	1 912 438 211 281 990 104
48	251 526 960 780 642 980 968	274 482 801 972 069 490	13 607 405 560 541 031 042
50	1 824 061 566 724 351 292 496	1 834 665 820 375 683 428	96 728 883 661 202 188 552
52	13 206 639 904 144 205 117 592	12 282 315 178 525 359 966	687 010 148 492 686 667 614
54	95 476 389 002 729 304 216 548	82 344 395 405 972 692 656	4 875 571 799 890 192 459 056
56	689 283 065 294 740 945 143 208	552 806 313 387 704 627 982	34 575 571 741 149 137 524 846
58	4 969 805 963 839 723 557 919 424	3 715 834 986 939 390 916 244	245 029 144 855 912 573 003 776
60	35 789 811 145 967 164 348 552 960	25 006 203 000 374 020 526 746	1 735 367 234 605 432 029 439 794
62	257 449 325 423 816 274 956 954 508	168 466 668 960 946 012 707 912	12 283 126 555 855 361 655 011 856
64	1 849 981 836 861 769 186 990 365 288	1 136 122 707 072 612 282 498 874	86 893 466 632 100 569 644 163 186
66	13 280 506 839 637 150 191 613 774 736	7 669 275 741 518 968 346 891 172	614 385 797 629 196 735 502 076 968
68	95 248 670 945 282 200 958 664 147 712	51 817 515 409 677 258 092 083 006	4 341 950 222 145 487 318 409 546 446
70	682 533 032 784 692 897 614 712 920 788	350 404 221 555 935 013 278 573 224	30 671 194 434 233 707 728 683 946 784
72	4 886 864 684 580 008 620 898 035 643 960	2 371 438 542 131 929 578 320 200 646	216 565 948 566 766 116 053 230 547 838
74	34 962 179 240 623 880 562 564 354 461 036	16 061 466 455 829 089 444 235 194 204	1 528 529 336 761 773 075 102 657 075 616
76	249 946 063 483 045 736 235 271 147 799 248	108 860 864 860 439 323 866 007 261 128	10 784 279 532 727 353 410 458 586 600 848
78	1 785 625 611 982 607 482 936 563 853 493 112	738 338 427 155 234 332 385 671 368 928	76 059 086 282 576 056 911 156 299 311 952
80	12 748 122 227 351 375 676 612 377 672 210 416	5 010 964 557 143 508 508 512 736 679 936	536 243 262 589 039 476 652 829 061 618 528
82	90 955 298 658 999 234 326 739 061 737 970 500	34 029 495 976 431 151 261 075 225 822 320	3779 470 144 925 357 385 934 811 283 997 288

3. Analysis of the series

The series listed in table 1 have coefficients which grow exponentially, with sub-dominant term given by a critical exponent. The generic behaviour is $G(u) = \sum_n g_n u^n \sim (1 - u/u_c)^{-\xi}$, and hence the coefficients of the generating function $g_n \sim \mu^n n^{\xi-1}$, where $\mu = 1/u_c$. To obtain the singularity structure of the generating functions we used the numerical method of differential approximants [17]. In particular, we used this method to estimate the critical exponents (we already have very accurate estimates for u_c from [8]). Since all odd terms in the series are zero and the first non-zero term is g_4 we actually analysed the function $F(u) = \sum_n g_{2n+4} u^n$. Combining the relationship given above between the coefficients in a series and the critical behaviour of the corresponding generating function with the expected behaviour (1) of the mean-square radius of gyration and moments of area yields the following prediction for their generating functions:

$$\mathcal{R}_g^2(u) = \sum_n p_{2n+4} \langle R^2 \rangle_{2n+4} (2n+4)^2 u^n = \sum_n r_n u^n \sim R(u)(1 - u\mu^2)^{-(\alpha+2\nu)} \tag{9}$$

$$\mathcal{P}^{(k)}(u) = \sum_n p_{2n+4} \langle a^k \rangle_{2n+4} u^n = \sum_n a_n^{(k)} u^n \sim a^{(k)}(u)(1 - u\mu^2)^{2-(\alpha+2k\nu)}. \tag{10}$$

Thus we expect these series to have a critical point, $u_c = 1/\mu^2 = 0.143\,680\,629\,27(1)$, known to a very high degree of accuracy from the analysis in [8], and as stated previously the exponent $\alpha = \frac{1}{2}$, while it is expected that $\nu = \frac{3}{4}$.

Estimates of the critical point and critical exponents were obtained by averaging values obtained from second-order $[L/N; M; K]$ inhomogeneous differential approximants. For each order L of the inhomogeneous polynomial we averaged over those approximants to the series which used at least the first 80–90% of the terms of the series. We used only approximants where the difference between N, M, and K did not exceed 2. Some approximants were excluded from the averages because the estimates were obviously spurious. The error quoted for these estimates reflects the spread (basically one standard deviation) among the approximants. Note that these error bounds should *not* be viewed as a measure of the true error as they cannot include possible systematic sources of error. In table 2 we have listed the results of our analysis. It is evident that the estimates for u_c and the critical exponents are in agreement with the expected behaviour. There are only some minor discrepancies in the fourth digit between the conjectured exponents and the estimates. This discrepancy is readily resolved by looking at the evidence in figure 1, where we have plotted the estimates for the critical point and exponent of $\mathcal{R}_g^2$. Each point in these figures represent an estimate obtained from a specific second order differential approximant with the various points obtained by varying the order of the polynomials in the approximants. It is clear that the estimates have not yet settled down to their asymptotic values and that they do converge towards the expected values as the number of terms used by the approximants is increased.

Now that the exact values of the exponents has been confirmed we turn our attention to the 'fine structure' of the asymptotic form of the coefficients. In particular, we are interested in obtaining accurate estimates for the amplitudes B, D and $E^{(1)}$. We do this by fitting the coefficients to the assumed form (1).

The asymptotic form of the coefficients p_n of the polygon generating function has been studied in detail previously [8, 18]. As argued in [18] there is no sign of non-analytic corrections-to-scaling exponents to the polygon generating function and one therefore finds that

$$p_n = \mu^n n^{-5/2} \sum_{k=0} a_k / n^k. \tag{11}$$

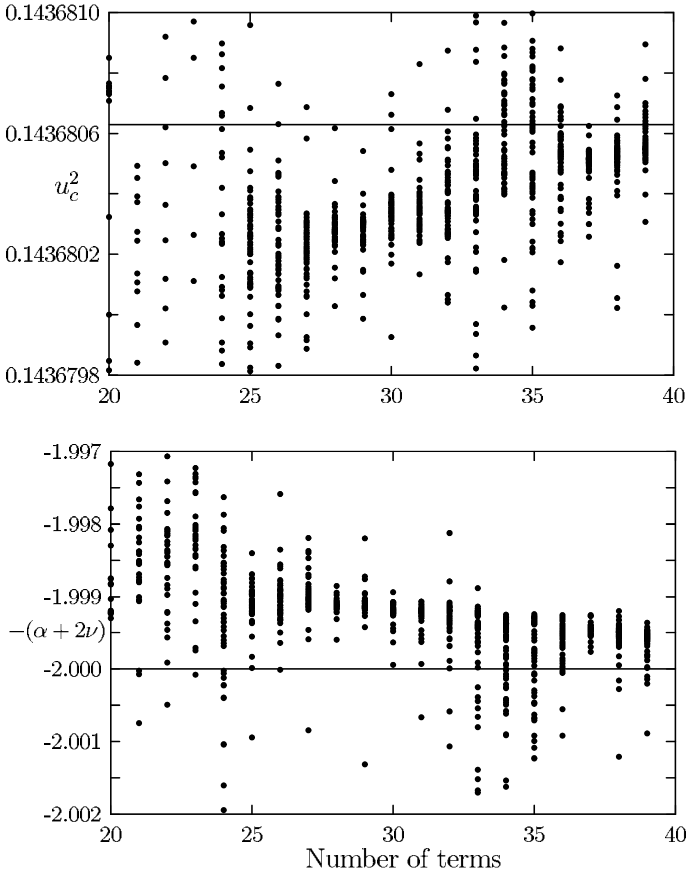

Figure 1. Estimates for the critical point and exponent of the generating function for the radius of gyration of square lattice polygons as a function of the number of terms used by the second-order differential approximants. The solid lines indicate the expected values $u_c = 0.143\,680\,629\,27(1)$ and $\xi = -(\alpha + 2\nu) = -2$.

Table 2. Estimates for the critical point u_c and exponents obtained from second-order differential approximants to the series for the radius of gyration, first and second moments of area of square lattice SAPs. L is the order of the inhomogeneous polynomial.

Series:	$\mathcal{R}_g^2(u)$		$\mathcal{P}^{(1)}(u)$		$\mathcal{P}^{(2)}(u)$	
L	u_c	$-(\alpha+2\nu)$	u_c	$2-(\alpha+2\nu)$	u_c	$2-(\alpha+4\nu)$
0	0.143 680 45(10)	−1.999 41(23)	0.143 680 543(86)	0.000 27(19)	0.143 680 502(35)	−1.499 523(78)
1	0.143 680 57(14)	−1.999 76(54)	0.143 680 556(33)	0.000 279(78)	0.143 680 539(19)	−1.499 662(62)
2	0.143 680 63(13)	−1.999 97(58)	0.143 680 558(31)	0.000 266(66)	0.143 680 535(22)	−1.499 658(86)
3	0.143 680 48(11)	−1.999 48(21)	0.143 680 562(25)	0.000 267(48)	0.143 680 530(20)	−1.499 58(21)
4	0.143 680 540(71)	−1.999 56(18)	0.143 680 567(25)	0.000 253(57)	0.143 680 541(21)	−1.499 636(57)
5	0.143 680 553(60)	−1.999 59(18)	0.143 680 566(27)	0.000 254(66)	0.143 680 545(15)	−1.499 654(28)
6	0.143 680 542(33)	−1.999 544(92)	0.143 680 577(13)	0.000 18(17)	0.143 680 542(15)	−1.499 649(58)
7	0.143 680 46(10)	−1.999 42(14)	0.143 680 564(23)	0.000 255(58)	0.143 680 542(16)	−1.499 658(64)
8	0.143 680 511(43)	−1.999 474(96)	0.143 680 568(21)	0.000 246(48)	0.143 680 539(17)	−1.499 650(57)
9	0.143 680 527(64)	−1.999 52(15)	0.143 680 582 8(92)	0.000 214(30)	0.143 680 548(28)	−1.499 674(71)
10	0.143 680 511(41)	−1.999 472(94)	0.143 680 572(17)	0.000 238(43)	0.143 680 544(16)	−1.499 665(52)

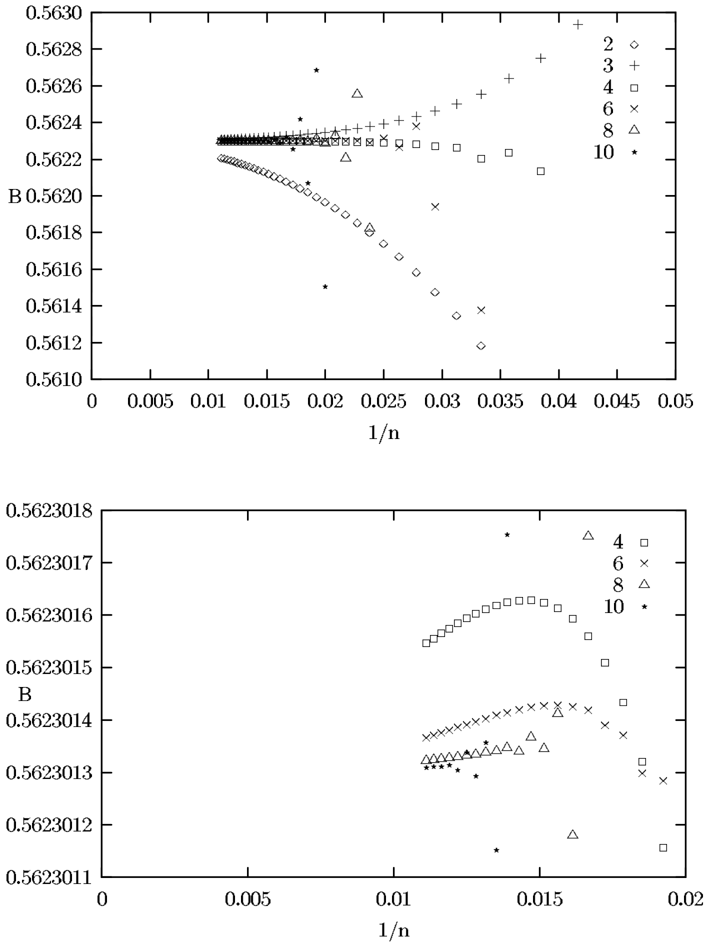

Figure 2. Estimates for the leading amplitude $a_0 = B$ of square lattice polygons as a function of $1/n$. Each data set is obtained by fitting p_n to the form given in equation (11) using from 2 to 10 correction terms. The lower panel displays a detailed look at the data in the upper panel.

This form was confirmed with great accuracy in [8]. Estimates for the leading amplitude $B = a_0$ can thus be obtained by fitting p_n to the form given in equation (11). In order to check the behaviour of such estimates we did the fitting using from 2 to 10 terms in the expansion. The results for the leading amplitude are displayed in figure 2. We notice that all fits appear to converge to the same value as $n \to \infty$, and that, as more and more correction terms are added to the fits the estimates exhibits less curvature and that the slope become smaller (although the fits using ten terms are a little inconclusive). This is very strong evidence that (11) indeed is the correct asymptotic form of p_n. We estimate that $B = 0.562\,301\,2(1)$.

The asymptotic form of the coefficients r_n in the generating function for the radius of gyration has not been studied previously. When fitting to a form similar to equation (11), assuming that here are only analytic corrections-to-scaling, we find that the amplitudes of higher order terms are very large and that the leading amplitude converge rather slowly. This indicates that this asymptotic form is incorrect. We find that the coefficients fit better if we assume a leading non-analytic correction-to-scaling exponent $\Delta = \frac{3}{2}$. This result confirms

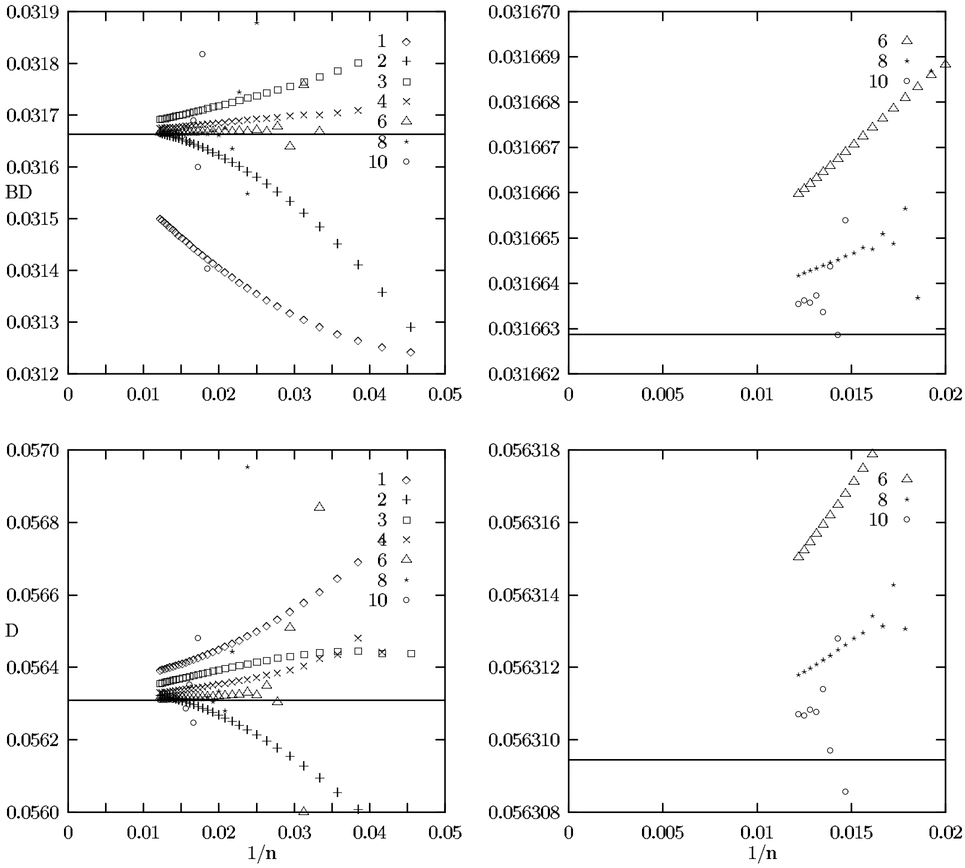

Figure 3. Estimates for the leading amplitude BD and D of the radius of gyration of square lattice polygons as a function of $1/n$. Each data set in the top panels is obtained by fitting the coefficients r_n of the radius of gyration generating function to the form given in equation (12), using from 1 to 10 correction terms. Each data set in the bottom panels is obtained by fitting r_n/p_n to the form given in equation (13). The right panels are a detailed look at the data in the left panels. The solid lines indicate the expected values given in the text.

the prediction of Nienhuis [14]. Note, that since the polygon generating function exponent $2-\alpha = \frac{3}{2}$ a correction-to-scaling exponent $\Delta = \frac{3}{2}$ is perfectly consistent with the asymptotic form (11). Because $2-\alpha+\Delta$ is an integer the non-analytic correction term becomes part of the analytic background term [18]. We thus propose the following asymptotic form:

$$r_n = \mu^n n \left[BD + \sum_{k=0} a_k/n^{k/2} \right]. \tag{12}$$

Alternative we could fit to the form

$$r_n/p_n = n^{7/2} \left[D + n^{5/2} \sum_{k=0} a_k/n^{k/2} \right]. \tag{13}$$

In figure 3 we show the leading amplitudes resulting from such fits while using from 1 to 10 terms in these expansions. Also shown in these figures (solid lines) are the predicted exact value of BD, given in equation (2), and the prediction for D using the estimate for B obtained above. As can be seen the leading amplitudes clearly converge towards their expected values and from these plots we can conclude that the prediction for BD has been confirmed to at least six digit accuracy. Assuming that equation (2) is exact and using the very accurate estimate for B we find that $D = 0.056\,309\,44(1)$.

Fitting the coefficients for the area-weighted moments to asymptotic forms similar to equations (12) and (13) above (only the leading exponent was changed accordingly) leads to the estimates $E^{(1)} = 0.141\,520(1)$ and $E^{(2)} = 0.021\,250\,5(4)$.

As stated above, the analysis of the polygon generating function is fully consistent with the prediction $\Delta = \frac{3}{2}$. However, all one can conclude from the analysis is that, *if* non-analytic correction-to-scaling terms are present, the exponents have to be 'half-integer', so that the correction terms become part of the analytic background. The detailed analysis of the asymptotic form of the coefficients in the generating functions for the radius of gyration and area-weighted moments provide the firmest evidence to date for the *actual existence* of a leading non-analytic correction to scaling exponent $\Delta = \frac{3}{2}$, thus confirming the theoretical predictions made by Nienhuis [14].

4. Conclusion

We have presented an improved algorithm for the calculation of the radius of gyration and area-weighted moments of SAPs on the square lattice. This algorithm has enabled us to calculate these series for polygons up to perimeter length 82. Our extended series enables us to give very precise estimate of the critical exponents, which are consistent with the exact values $\alpha = \frac{1}{2}$ and $\nu = \frac{3}{4}$. We also obtain a very precise estimate for the amplitude $B = 0.562\,301\,2(1)$. Analysis of the coefficients of the radius of gyration series yielded results fully compatible with the prediction $BD = 5/16\pi^2$. This allows us to obtain the very accurate estimate $D = 0.056\,309\,44(1)$. From the first area-weighted moment we obtained the estimate $E^{(1)} = 0.141\,520(2)$, which allows us to give a much improved estimate for the universal amplitude ratio $E^{(1)}/D = 2.513\,26(3)$. We also find firm evidence for the existence of a non-analytic correction-to-scaling term with exponent $\Delta = \frac{3}{2}$.

E-mail or WWW retrieval of series

The series for the various generating functions studied in this paper can be obtained via e-mail by sending a request to I.Jensen@ms.unimelb.edu.au or via the world wide web on the URL http://www.ms.unimelb.edu.au/~iwan/ by following the instructions.

Acknowledgments

Thanks to Tony Guttmann for many valuable comments on the manuscript and the series analysis. Financial support from the Australian Research Council is gratefully acknowledged.

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